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QUARTERING THE AREA OF SQUARES USING VERTICALLY TRANSLATED PARALLEL LINES

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Abstract. Any line passing through the center of a square will partition the square into two halves with equal area. By drawing two more lines, each parallel with and equidistant to the center line, we can partition the square into four quarters with equal area. The vertical distance between the center line and each flanking line which is required to quarter the square can be expressed as a function p of the angle θ of the center line to the sides of the square. This can be initially demonstrated for a unit square, and then be extended to work for all squares. In this paper, we seek to explain the function, to prove its validity, and to explore its applications.

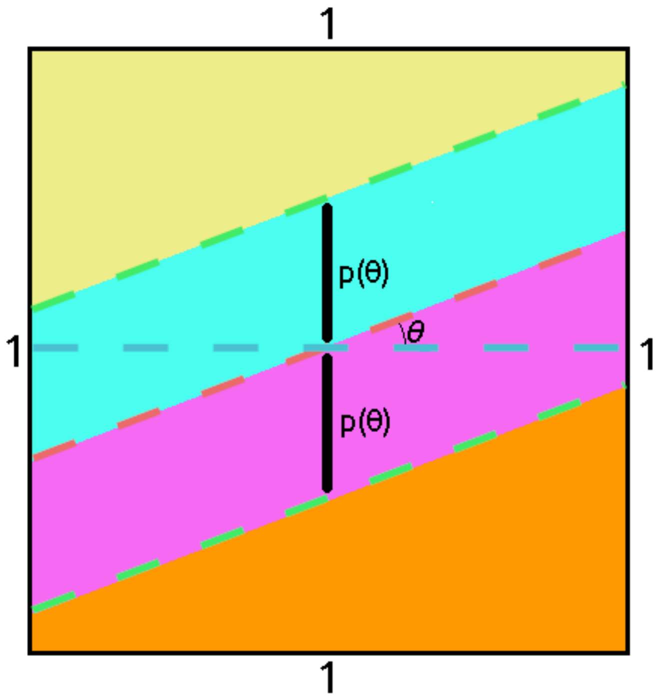

FIGURE 1. Square partitioned by 3 parallel lines into 4 sections with equal area.

1. Introduction

The topics covered in this paper will be fairly elementary and require little more than a high school level of mathematics understanding. Most of the paper is rooted in trigonometry. Little of what is covered requires a formal introduction. One possible exception to this is the unit square, a square with side lengths and area 1.

Later in the paper, we'll explore different squares on a cartesian plane. To describe these squares, we will be using a unique center-scale-angle (CSA) representation. Each square will be described using the cartesian coordinates of its center, the lengths of its sides, and the angle of the square's "equator" (a line passing through the center of the square, parallel to one pair of its bases) relative to the x-axis. In a CSA system, a square of area one, centered on the origin, the sides of which are parallel to the axes of the plane, could be described as

$$c_x = 0, c_y = 0, s = 1, a = 0$$

where c_x and c_y describe the center coordinates, s describes the side length, and a describes the angle of the square to the x-axis. With this system, all squares on a Cartesian plane can be described in a manner which makes them much easier to partition into quarters.

In this paper, we'll first describe the situation that our equation addresses, and then the Square Parallel Quartering Function itself. We'll then prove that the equation is accurate. Next, we'll extend the equation to all squares on a Cartesian Plane. Lastly, we will discuss the potential applications of this exciting and accessible formula.

2. Findings

Any line drawn through the center of a square will partition the square into two sections of equal area [1]. It does not matter the size, rotation, or position of the square, the line will partition the square all the same. Imagine a square of area and side length 1, centered on the origin, with its sides parallel to the plane's axes.

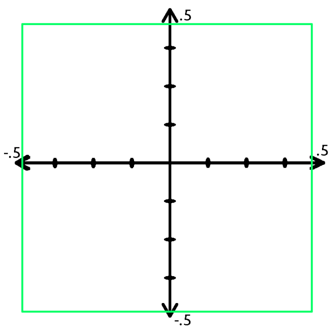

FIGURE 2. A unit square (square of side length and area 1) centered on the origin.

Any line we draw through the origin here will divide the square into two parts, each with an equal area (in this case, an area of $\frac{1}{2}$.) We'll call this the *center line*. The center line is not fixed, and can be rotated to any position within the square. We'll represent the position of the center line using its angle θ to the x-axis.

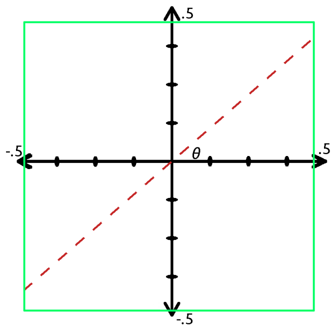

FIGURE 3. Unit square, transited by one line through its center.

Let's also add two more lines, each parallel with and equally distant from the center line, and which split the square into four quarters of equal area. We'll call these new lines "flanking lines." Remember, angle θ is arbitrary.

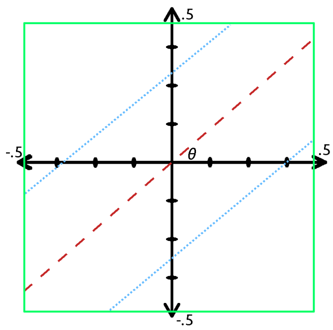

FIGURE 4. Three parallel lines divide a square into four quarters.

We now have three parallel lines which divide the square into four quarters, and which can be rotated to represent any arbitrary angle. But what is the distance between the lines? The answer depends on what angle θ is. We can express the distance as a function, $p(\theta)$, where p is a piecewise function:

$$p(\theta) = \begin{cases} \frac{1}{4} & \textit{if } 0 \leq |\tan(\theta)| < \frac{1}{2} \\ \frac{1}{2} - |\tan(\theta)| \cdot (\sqrt{|\frac{\cot(\theta)}{2}|} - \frac{1}{2}) & \textit{if } \frac{1}{2} \leq |\tan(\theta)| < 2 \\ |\frac{\tan(\theta)}{4}| & \textit{if } 2 \leq |\tan(\theta)| \end{cases}$$

The vertical distance between the flanking lines and the center line is described by $p(\theta)$, at all values except $\theta = \frac{\pi}{2}$. At this value, the function is undefined, since a vertical line cannot be vertically translated to produce anything other than itself.

Since the center line is a linear function, we can call it $c(x)$ and define it as $x\tan(\theta)$. The higher flanking line can then be defined as:

$$f_h(x) = c(x) + p(\theta)$$

While the lower can be defined as:

$$f_l(x) = c(x) - p(\theta)$$

These linear functions provide an adequate model to describe a square which has been quartered by three parallel lines. Interestingly, each piece of the piecewise function $p(\theta)$ generates differently shaped sections.

The first piece produces four vertical bands, two quadrilaterals on top and bottom, and two parallelograms in the middle. This occurs when the center line and both flanking lines intersect the sides which are parallel to the y-axis, over the period where $|\tan(\theta)| < \frac{1}{4}$.

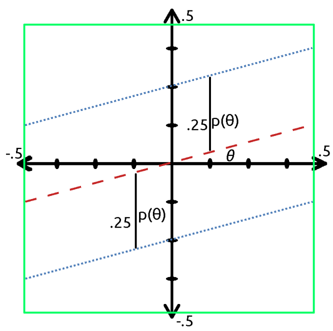

FIGURE 5. Horizontally quartered square

The second piece produces four diagonal bands, with two triangles in opposite corners, and two pentagons in the middle. The corners which the triangles appear in are determined by the precise value of θ. If $\tan(\theta) > 0$, then the triangles will be in the top left and bottom right corners of the square. If $\tan(\theta) < 0$, however, then the triangles will appear in the top right and bottom left corners of the square. This occurs when one flanking line intersects the sides of the square parallel to the x-axis while the other intersects the sides of the square parallel to the y-axis.

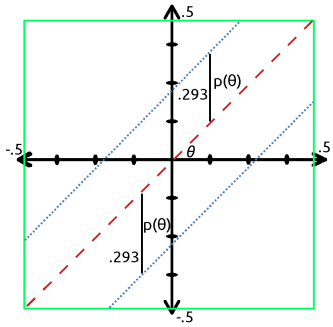

FIGURE 6. Diagonally quartered square

The third piece is much like the first in that there are two inlying parallelogram bands surrounded by two outlying quadrilateral bands. The difference is that whereas the bands of the first piece are separated vertically, the bands of the third are separated horizontally. This pattern occurs when both flanking lines intersect sides of the square parallel to the x-axis.

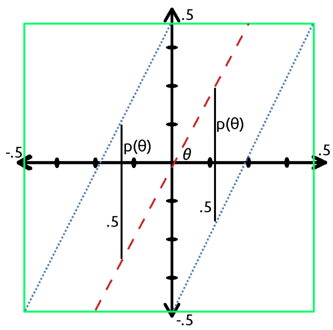

FIGURE 7. Vertically quartered square

Overall, the parallel square quartering function $p(\theta)$ provides the necessary distance between three parallel lines to make them divide a unit square into four quarters of equal area, based on the angle θ of the three parallel lines to the sides of the unit square parallel to the x-axis. The function $p(\theta)$ is a piecewise function which generates three different shapes, and the function works at all values of θ except for $\theta = \frac{1}{4}$. In the following sections, we will prove each piece of the function, extend the function to all squares on a coordinate plane, and explore the function's applications.

3. Proof

Since the parallel square quartering function is in three pieces which generate three different shapes, the proof of the function must too be in three pieces. Just like in the preceding section, we will cover the vertical arrangement, then the diagonal, and lastly the horizontal.

3.1. Vertical Piece. The first piece we will look at will be the vertical piece. Since the vertical piece has a constant value of $\frac{1}{4}$, it'll be the simplest piece to prove. The center line intersects the horizontal sides of the square when $|c(\frac{1}{2})| < \frac{1}{2}$. Since we know that $c(x) = x\tan(\theta)$, we can substitute for the function in the inequality:

$$|x\tan(\theta)| < \frac{1}{2}$$

With some simple algebra and trigonometry, we can bring this to a more friendly form.

$$|\frac{1}{2}\tan(\theta)| < \frac{1}{2}$$
$$\frac{1}{2}|\tan(\theta)| < \frac{1}{2}$$
$$|\tan(\theta)| < 1$$

So, the center line intersects the sides of the square parallel to the y-axis for values of θ where $|\tan(\theta)| < 1$, which includes the subset where $|\tan(\theta)| < \frac{1}{2}$. What about the flanking lines? Well, the story is similar for the flanking lines, but their vertical position is translated up and down an additional $\frac{1}{4}$. To prove that the flanking lines intersect the sides of the square parallel to the y-axis, we can simply do a fundamentally identical series of equations:

$$\frac{1}{2}|\tan(\theta)| + \frac{1}{4} < \frac{1}{2}$$
$$\frac{1}{2}|\tan(\theta)| < \frac{1}{4}$$
$$|\tan(\theta)| < \frac{1}{2}$$

So, the center line and flanking lines intersect vertical sides of the square for values of θ where $|\tan(\theta)| < \frac{1}{4}$. Since $|\tan(\theta)| < \frac{1}{2}$ is more restrictive than $|\tan(\theta)| < 1$, we'll be referring to the former for now. Where $|\tan(\theta) < \frac{1}{2}$, both the center line and flanking lines intersect the sides of the square parallel to the y-axis. Since the sides of the square are parallel to each other, as is each pair of center line and flanking line, this shape is two intersecting pairs of parallel lines: a parallelogram.

The next step to prove the validity of the first piece is to prove that these parallelograms really do have an area of $\frac{1}{4}$. To do so is actually quite easy! The area of a parallelogram can be expressed as base times height. Let's say that the base is parallel to the x-axis, and the height is parallel to the y-axis. The x-axis intersects the two vertical sides of the square, which are a distance of 1 apart from each other. Therefore, the base is 1. Since $p(\theta)$ for the first piece is $\frac{1}{4}$, the height should be $\frac{1}{4}$. Together, this produces a parallelogram with an area of $\frac{1}{4}$. Since the center line splits the square into two parts of equal area [the other quadrilateral on each side must carry the remainder of the area, that being $\frac{1}{4}$. In summary: the first piece of the parallel square quartering function accurately describes the necessary distance between parallel lines to divide a square into four quarters of equal area for values of θ where $|\tan(\theta)| < \frac{1}{2}$.

3.2. Diagonal Piece. The "diagonal piece" of the square quartering function, effective for values of θ where $\frac{1}{2} \leq |\tan(\theta)| \leq 2$, is probably the most difficult piece of the function to prove, so we will take a different route from the proofs for the other sections. Since the middle sections, which we examine in the other proofs, are no longer parallelograms, but rather pentagons, there is no easy proof for the area of the middle section. Instead, we will examine the triangle sections outside of the flanking lines. We will first prove that the flanking lines do indeed create triangles, and then will prove that those triangles have an area of $\frac{1}{4}$.

Through the diagonal piece, each flanking line creates a right triangle, itself being the hypotenuse of the square, and two perpendicular sides of the square being the square's legs. To prove this, we must take on an algebraic view of intersection. In a unit square centered on the origin, the horizontal sides of the square can be represented algebraically as $y = \pm\frac{1}{2}\{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$. To intersect these lines, the value of the flanking lines must equal $\pm\frac{1}{2}$ within the interval $[-\frac{1}{2}, \frac{1}{2}]$. On the other hand, the algebraic equations of the vertical sides of the square can be represented as $x = \pm\frac{1}{2}\{-\frac{1}{2} \leq y \leq -\frac{1}{2}\}$. To intersect these sides, the values of the flanking lines must fall in the interval of $[-\frac{1}{2}, \frac{1}{2}]$ where $x = \pm\frac{1}{2}$.

Let's start by proving that the flanking lines each intersect the horizontal sides of the square. This would mean that f_h, the higher flanking line, must intersect $y = \frac{1}{2}$, and f_l intersect $y = -\frac{1}{2}$, over the period $(-\frac{1}{2}, \frac{1}{2})$. This is fairly complicated, and not something we're interested in expressing in full. By examining some geometric properties of the shape, we can express this in a much simpler way.

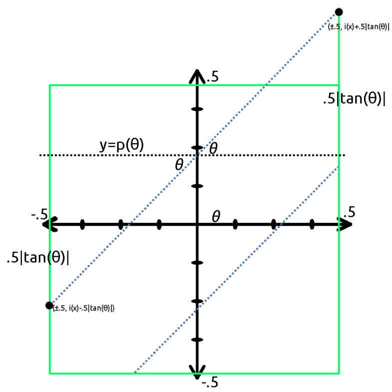

FIGURE 8. Example of the equality of $|f(\frac{1}{2}) - f(0)|$ and $|f(-\frac{1}{2}) - f(0)|$

If we draw a horizontal line which intersects each flanking line at the y-axis, as well as true lines at $x = \pm\frac{1}{2}$, we can create two right triangles. Because all horizontal lines are parallel, and because the flanking lines are parallel to the center line, the angle created by their intersect will be equal to the angle of the intersection of the center line and x-axis, written as θ. We can use this information to construct a triangle.

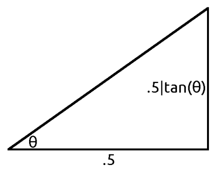

FIGURE 9. Triangle of a flanking line, $x = \pm\frac{1}{2}$, and $y = \pm\frac{1}{2}$.

Using this triangle, we know that the absolute difference between the value of a flanking line at $x = 0$ and the value at $|x| = \frac{1}{2}$ is $\frac{1}{2}|\tan(\theta)|$. We can call this difference d. Using d, we can greatly simplify our proof of the triangular shape of the diagonal piece.

The value of the flanking line with the greater absolute value will be the value at which the addition expression is used, since $|a + b| > |a - b|$. Therefore, the side with the addition will be the side which lies outside of the square. Using this, we know that the flanking line will intersect $x = \pm\frac{1}{2}$ at $\pm\frac{1}{2}|\tan(\theta)| + p(\theta)$. So, for a flanking line to cross the horizontal lines, it must be the case that $\frac{1}{2}|\tan(\theta)| + p(\theta) \geq \frac{1}{2}$. Using algebra, we can prove that this is the case for all values of θ where $\frac{1}{2} \leq |\tan(\theta)| \leq 2$:

$$\frac{1}{2}|\tan(\theta)| + p(\theta) \geq \frac{1}{2}$$
$$\frac{1}{2}|\tan(\theta)| + \frac{1}{2} - |\tan(\theta)|(\sqrt{\frac{|\cot(\theta)|}{2}} - \frac{1}{2}) \geq \frac{1}{2}$$
$$\frac{1}{2}|\tan(\theta)| + \frac{1}{2} - (|\tan(\theta)|\sqrt{\frac{|\cot(\theta)|}{2}} - \frac{1}{2}|\tan(\theta)|) \geq \frac{1}{2}$$
$$\frac{1}{2}|\tan(\theta)| + \frac{1}{2} - |\tan(\theta)|\sqrt{\frac{1}{2}|\cot(\theta)|} + \frac{1}{2}|\tan(\theta)| \geq \frac{1}{2}$$
$$|\tan(\theta)| + \frac{1}{2} - |\tan(\theta)|\sqrt{\frac{1}{2}|\cot(\theta)|} \geq \frac{1}{2}$$
$$|\tan(\theta)| - |\tan(\theta)|\sqrt{\frac{|\cot(\theta)|}{2}} \geq 0$$
$$|\tan(\theta)|(1 - \sqrt{\frac{|\cot(\theta)|}{2}}) \geq 0$$
$$1 - \sqrt{\frac{|\cot(\theta)|}{2}} \geq 0$$
$$1 \geq \sqrt{\frac{|\cot(\theta)|}{2}}$$
$$1 \geq \frac{1}{2}|\cot(\theta)|$$
$$2 \geq |\cot(\theta)|$$
$$\frac{1}{2} \leq |\tan(\theta)|$$

Therefore, the flanking line intersects a horizontal side of the square for values of θ where $\frac{1}{2} \leq |\tan(\theta)|$, including the subset where $\frac{1}{2} \leq |\tan(\theta)| \leq 2$.

Now that we have the principles covered, proving the vertical intersection is much simpler. Since the vertical sides of the square are defined as $x = \pm\frac{1}{2}\{-\frac{1}{2} \leq x \leq \frac{1}{2}\}$, the flanking line must have a value at either $x = \frac{1}{2}$ or $x = -\frac{1}{2}$ in the interval $(-\frac{1}{2}, \frac{1}{2})$. We know from the previous paragraphs that the intersect of the flanking line and $x = \pm\frac{1}{2}$ outside of the square will have a value of $p(\theta) + d$. By the same token, we know that the value of the flanking line within the square will have a value of $p(\theta) - d$. So, to intersect a vertical side of the square, it must be the case that $p(\theta) - \frac{1}{2}|\tan(\theta)| \geq -\frac{1}{2}$. With some algebra, we can again prove that this statement is true for all values of $\frac{1}{2} \leq |\tan(\theta)| \leq 2$:

$$
\begin{gathered}
p(\theta) - \frac{1}{2}|\tan(\theta)| \geq -\frac{1}{2} \\
\frac{1}{2} - (\sqrt{\frac{1}{2}|\cot(\theta)|} - \frac{1}{2})|\tan(\theta)| - \frac{1}{2}|\tan(\theta)| \geq -\frac{1}{2} \\
\frac{1}{2} - (|\tan(\theta)|\sqrt{\frac{1}{2}|\cot(\theta)|} - \frac{1}{2}|\tan(\theta)|) - \frac{1}{2}|\tan(\theta)| \geq -\frac{1}{2} \\
\frac{1}{2} - |\tan(\theta)|\sqrt{\frac{1}{2}|\cot(\theta)|} + \frac{1}{2}|\tan(\theta)| - \frac{1}{2}|\tan(\theta)| \geq -\frac{1}{2} \\
\frac{1}{2} - |\tan(\theta)|\sqrt{\frac{1}{2}|\cot(\theta)|} \geq -\frac{1}{2} \\
-|\tan(\theta)|\sqrt{\frac{1}{2}|\cot(\theta)|} \geq -1 \\
|\tan(\theta)|\sqrt{\frac{1}{2}|\cot(\theta)|} \leq 1 \\
\frac{1}{2}|\tan^2(\theta)||\cot(\theta)| \leq 1 \\
\frac{1}{2}|\tan(\theta)| \leq 1 \\
|\tan(\theta)| \leq 2
\end{gathered}
$$

Therefore, the flanking lines intersect a vertical side of the square for values of θ where $|\tan(\theta)| \leq 2$, including the subset where $\frac{1}{2} \leq |\tan(\theta)| \leq 2$.

Since the flanking lines, while $\frac{1}{2} \leq |\tan(\theta)| \leq 2$, each intersect a vertical line and a horizontal line, they each form a triangle. The next (and final) step of our proof is to prove that these triangles have an area of $\frac{1}{4}$. To do this, we must write an equation to describe the area of the triangle. The equation for the area of a triangle is $\frac{B \cdot H}{2}$, where B is the base, and H is the height of the triangle. So, we must find the base and height of the triangle.

Finding the height of the triangle is fairly simple. The height of the triangle will be along the vertical side of the square, between the intersect of that vertical side with a horizontal side and the intersect of the vertical side with the flanking line. Let's look at a model.

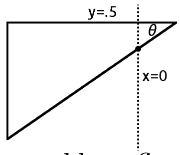

FIGURE 10. Triangle formed by a flanking line's intersections with the sides of the square.

In the model, we can see the height is the sum of the difference between the flanking line at $x = 0$ and the horizontal line at $y = \frac{1}{2}$, and d, the change in the y-value of the flanking line over $\frac{1}{2}$. We can, therefore, write the height of the triangle as $\frac{1}{2}|\tan(\theta)| + (\frac{1}{2} - a)$.

Based on the triangle, we know that the length of the base of the triangle is equal to the length of the height times $|\cot(\theta)|$. To find base times height then, we can just square height and multiply by $|\cot(\theta)|$. So, we get our equation for the area of the triangle: $\frac{(\frac{1}{2}|\tan(\theta)|+(\frac{1}{2}-i(x))^2\cdot|\cot(\theta)|}{2}$. Using algebra, we can prove that the area of the triangle equals $\frac{1}{4}$.

$$\frac{\left(\frac{1}{2}\left|\tan\left(\theta\right)\right| + \left(\frac{1}{2} - p(\theta)\right)\right)^2 \cdot \left|\cot\left(\theta\right)\right|}{2} = \frac{1}{4}$$

$$\frac{\left(\frac{1}{2}\left|\tan\left(\theta\right)\right| + \left(\frac{1}{2} - \left(\frac{1}{2} - \left|\tan\left(\theta\right)\right|\left(\sqrt{\frac{1}{2}\cot\left(\theta\right)} - \frac{1}{2}\right)\right)\right)\right)^2 \cdot \left|\cot\left(\theta\right)\right|}{2} = \frac{1}{4}$$

$$\frac{\left(\frac{1}{2}\left|\tan\left(\theta\right)\right| + \left(\frac{1}{2} - \left(\frac{1}{2} - \left|\tan\left(\theta\right)\right|\left(\sqrt{\frac{1}{2}\left|\cot\left(k\right)\right|} - \frac{1}{2}\right)\right)\right)\right)^2 \cdot \left|\cot\left(\theta\right)\right|}{2} = \frac{1}{4}$$

$$\frac{\left(\frac{1}{2}\left|\tan\left(\theta\right)\right| + \frac{1}{2} - \frac{1}{2} + \left|\tan\left(\theta\right)\right|\left(\sqrt{\frac{1}{2}\left|\cot\left(\theta\right)\right|} - \frac{1}{2}\right)\right)^2 \cdot \left|\cot\left(\theta\right)\right|}{2} = \frac{1}{4}$$

$$\frac{\left(\frac{1}{2}\left|\tan\left(\theta\right)\right| + \left|\tan\left(\theta\right)\right|\left(\sqrt{\frac{1}{2}\left|\cot\left(\theta\right)\right|} - \frac{1}{2}\right)\right)^2 \cdot \left|\cot\left(\theta\right)\right|}{2} = \frac{1}{4}$$

$$\frac{\left(\left(\frac{1}{2} + \left(\sqrt{\frac{1}{2}\left|\cot\left(\theta\right)\right|} - \frac{1}{2}\right)\right)\left|\tan\left(\theta\right)\right|\right)^2 \cdot \left|\cot\left(\theta\right)\right|}{2} = \frac{1}{4}$$

$$\frac{\left(\frac{1}{2}\left|\cot\left(\theta\right)\right|\left|\tan\left(\theta\right)\right|\right)^2 \cdot \left|\cot\left(\theta\right)\right|}{2} = \frac{1}{4}$$

$$\frac{\frac{1}{2}\left|\cot\left(\theta\right)\right| \cdot \left|\tan\left(\theta\right)\right|^2 \cdot \left|\cot\left(\theta\right)\right|}{2} = \frac{1}{4}$$

$$\frac{\frac{1}{2}\left|\cot\left(\theta\right)\right|^2 \cdot \left|\tan\left(\theta\right)\right|^2}{2} = \frac{1}{4}$$

$$\frac{\frac{1}{2}\cdot 1}{2} = \frac{1}{4}$$

$$\frac{1}{4} = \frac{1}{4}$$

So, for values of θ where $\frac{1}{2} \leq |\tan(\theta)| \leq 2$, the flanking lines form two triangles of area $\frac{1}{4}$. The remaining area is crossed by one straight line, so the remaining area must also consist of two sections of area $\frac{1}{4}$. Therefore, the square parallel quartering function is valid and true for values of θ where $\frac{1}{2} \leq |\tan(\theta)| \leq 2$.

3.3. Horizontal Piece. Think back to the proof of the vertical piece. The horizontal piece is much like the vertical piece, simply rotated. With the vertical piece, we examined intersections of the sides of the square parallel to the y-axis. Now, we must examine sides of the square parallel to the x-axis. We must prove that for values of θ where $|\tan(\theta)| > 2$, the center line and flanking lines intersect the sides of the square parallel to the x-axis. Thankfully, the requirements are exactly opposite to those for the vertical piece! For the vertical piece, each line needed to be less than $\frac{1}{2}$ at $x = \pm\frac{1}{2}$. For this horizontal piece, each line must be greater than $\frac{1}{2}$ at $x = \pm\frac{1}{2}$!

Just like with the vertical piece, let's start with the center line:

$$|c(\frac{1}{2})| > \frac{1}{2}$$
$$\frac{1}{2}|\tan(\theta)| > \frac{1}{2}$$
$$|\tan(\theta)| > 1$$

We can see that the center line intersects horizontal sides of the square for values of θ where $|\tan(\theta)| > 1$. Since the requirement of the horizontal piece to be operational is that $|\tan(\theta)| > 2$ and since $2 > 1$, the requirements of the function are satisfied.

Next, let's check the flanking lines. The absolute values of these flanking lines must too be greater than $\frac{1}{2}$ at x-value $\pm\frac{1}{2}$. Just like with the vertical piece, we're modifying the flanking lines by $p(\theta)$. We can also substitute the equation for $c(x)$ in for the function. After these substitutions, we're just a few steps of algebra away!

$$\frac{1}{2}|\tan(\theta)| - \frac{1}{4}|\tan(\theta)| > \frac{1}{2}$$
$$\frac{1}{4}|\tan(\theta)| > \frac{1}{2}$$
$$|\tan(\theta)| > 2$$

We've now showed that both the center lines and flanking lines intersect the sides of the square parallel to the x-axis for values of θ where $|\tan(\theta)| > 2$. Just like with the horizontal system, we know that the center line and flanking line are parallel to each other, as are the opposite sides of the square parallel to the x-axis. We therefore know that we have two pairs of intersecting parallel lines, so we have a parallelogram.

The final step is to prove that the area of this parallelogram really is $\frac{1}{4}$. Again, the formula for the area of a parallelogram is base times height. We'll again say that the base is parallel to the x-axis, and the height is parallel to the y-axis. The sides of the square parallel to the x-axis are separated by the y-axis at a distance of 1, so the height is 1. This means we must provide a base length of $\frac{1}{4}$. In other words, the flanking lines must be a horizontal distance of $\frac{1}{4}$ away from the center line. Algebraically, this is to say:

$$c(x - \frac{1}{4}) = f_l(x)$$

Expanding this into the individual functions:

$$\begin{aligned}
\tan(\theta)(x - \frac{1}{4}) &= c(x) - p(\theta) \\
x\tan(\theta) - \frac{\tan(\theta)}{4} &= c(x) - p(\theta) \\
c(x) - \frac{\tan(\theta)}{4} &= c(x) - p(\theta) \\
-\frac{\tan(\theta)}{4} &= -p(\theta) \\
\frac{\tan(\theta)}{4} &= p(\theta)
\end{aligned}$$

Since, for the horizontal piece, $p(\theta) = \frac{\tan(\theta)}{4}$, this proves that the shift of $p(\theta)$ really does provide a horizontal shift of $\frac{1}{4}$, meaning it creates a parallelogram with a width of $\frac{1}{4}$. Together with a height of 1, we have proven the area of the parallelogram to be $\frac{1}{4}$.

Like with the previous two parts, we know that the center line divides the square into two halves. Since the parallelograms take up one half of each half, the remaining quadrilaterals must each have an area of $\frac{1}{4}$. In summary, for values of θ where $|\tan(\theta)| > 2$, the parallel square quartering function accurately describes the vertical shift necessary to divide a unit square into equal quarters using only parallel lines. This, in combination with the other proofs for values of $|\tan(\theta)| < \frac{1}{2}$) and $\frac{1}{2} < |\tan(\theta)| < 2$ proves that the parallel square quartering function is true for all values of θ.

3.4. Proof Conclusion. In conclusion, we have proven that the parallel square quartering function $p(\theta)$ accurately describes the distance between three parallel lines necessary to divide a unit square into four quarters of equal area. We have proved it piecewise for each piece of the function, and we have proved it for the entire domain of the function (that being, all real numbers θ for which $\tan(\theta)$ is defined.

4. Extension to all Squares on a Cartesian Plane

The proofs explained in the last section, of course, only apply to a unit square centered on the origin. However, we can expand our model to accommodate all squares with relatively few changes.

Recall the CSA square model. We will be describing squares using the coordinates of their centers (c_x and c_y), the length of each side of the square (s), and the angle of one of the bases to the x-axis (a). These four values will give us our square.

Adjusting for the center is quite simple. Simply move the square such that it is centered on (c_x, c_y). Shift the center line function $c(x)$ horizontally so that it has a y-intercept at $(c_x, 0)$, then add c_y so that the center of the square falls on the center line. Since the positions of the flanking lines are dependent on the center line, these shifts to the center line will correctly reposition the flanking lines as well.

Adjusting for scale is also quite easy. The shift provided by $p(\theta)$ is perhaps best thought as a ratio of the needed shift to the side length of the square. When the sides of the square have a length of 1, $p(\theta)$ suffices. Otherwise, simply multiply each piece of $p(\theta)$ by the scale variable, s. This produces the necessary distance based on the size of the square and the angle between the center line and the sides of the square.

Adjusting $p(\theta)$ for the angle of the square to the x-axis is probably the most difficult adjustment to make. That said, it is not so difficult once you know what to do. First, we must rotate the center line. Any rotation of the square will be carried on to the center line, so we must redefine $c(x)$. This is quite simple.

$$c(x) = \tan(\theta + a)x$$

With this simple equation, we've fit in the center line with the rotation. Fitting in the flanking lines, however, may prove more difficult. Imagine that in a unit square, the shifts output by $p(\theta)$ are shifted along the y-axis. We'll call this the shift axis. The shift axis is perpendicular to angle a. As we rotate the square, we must also rotate the shift axis. The distance moved along the shift axis is constant for all values of a, but the representation on a cartesian plane is not. Thankfully, basic trig will carry us to a solution. To travel a distance of $p(\theta)$ along the shifting axis, we have to shift vertically by $\sin(a + \frac{\pi}{2})p(\theta)$ and horizontally by $\cos(a + \frac{\pi}{2})p(\theta)$. We can incorporate this into the functions for the flanking lines by redefining them:

$$f_h(x) = c(x - cos(a + \frac{\pi}{2})p(\theta)) + \sin(a + \frac{\pi}{2})p(\theta)$$
$$f_l(x) = c(x + cos(a + \frac{\pi}{2})p(\theta)) - \sin(a + \frac{\pi}{2})p(\theta)$$

With a little algebra, we can isolate c(x) so that it is independent from all other variables.

$$f_h(x) = c(x) - c(\cos(a + \frac{\pi}{2})p(\theta)) + \sin(a + \frac{\pi}{2})p(\theta)$$

We'll also add in s to account for the scale of the function, then substitute in the value of $c(x)$.

$$f_h(x) = c(x) - sc(\cos(a + \frac{\pi}{2})p(\theta)) + s \cdot \sin(a + \frac{\pi}{2})p(\theta))$$
$$f_h(x) = c(x) - s \cdot p(\theta) \cdot (\tan(\theta + a)\cos -(a + \frac{\pi}{2}) + \sin(a + \frac{\pi}{2})$$

We can consolidate the long term at the end into our new function, $m(x)$. For the line which we've explored, we subtract $m(x)$, but we will add $m(x)$ for the other line. We now have one cohesive multivariate function that accurately describes the vertical shift needed for all angles of θ, and for all squares on a coordinate plane!

5. Applications

The square parallel quartering function is a powerful function with many applications. The first application is in aesthetics, since making sections of squares with perfect equality may be of interest to some artists. Another application, in fact, the application that inspired this very formula, is in urban planning: if you wanted to plan two roads such that the furthest distance from each is one quarter of the total serviced area, this function may be your friend. The biggest application, though, is likely in education. This accessible equation may be useful in educating students on advanced questions of area and trigonometry.

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